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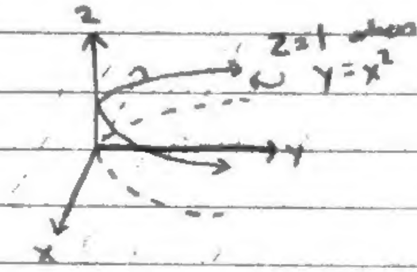
Last time: limits

recall: curves criterion: a function f has $\lim_{\vec{x}\to\vec{a}} f(x) = L$ iff for all continuous spaces curves $\vec{r}(t)$ with $\lim_{t\to 0^+} \vec{r}(t) = \vec{a}$, we have $\lim_{t\to 0^+} f(\vec{r}(t)) = L$

- To show a limit DNE, find 2 curves $r_0(t)$ & $r_1(t)$ with $\lim_{t\to 0^+} \vec{r}_i(t) =$ and show $\lim_{t\to 0^+} (r_0(t)) \neq \lim_{t\to 0^+} (r_1(t))$
- we used lines $l_{a,b}(t) = \vec{a} + t(a,b)$ last time...
  - These lines are not sufficient to show a line DNE

ex. Let $f(x,y) = \begin{cases} 1 & \text{if } y = x^2 \\ 0 & \text{otherwise} \end{cases}$



Limiting to $\vec{0}$ along the lines $l_{a,b}(t)$ we notice

$f(l_{a,b}(t)) = f(at, bt) = 0$ for all $t > 0$, except at most 1 value of t (b/c $(at)^2 = bt$ has at least 2 solutions)

$\therefore \lim_{t\to 0^+} f(l_{a,b}(t)) = \lim_{t\to 0^+} 0 = 0$

On the other hand, limiting along $\vec{r}(t) = \langle t, t^2 \rangle$, we see:

$f(\vec{r}(t)) = f(t, t^2) = 1$ for all t. Hence, $\lim_{t\to 0^+} f(\vec{r}(t)) = \lim_{t\to 0^+} 1 = 1$

- since $1 \neq 0$, we see $\lim_{\vec{x}\to\vec{0}} f(\vec{x})$ DNE by the curves criterion //

Question: how do we show that a limit does exist?

Trick: Use polar coordinates (works sometimes)

ex. does $\lim_{\vec{x}\to\vec{0}} \frac{\sin(x^2+y^2)}{x^2+y^2}$ exist?

sol. First, convert to polar coordinates: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$(x,y) \to (0,0)$ iff $r \to 0^+$

if it exists

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0^+} \frac{\sin((r\cos\theta)^2+(r\sin\theta)^2)}{(r\cos\theta)^2+(r\sin\theta)^2}$$

$$= \lim_{r\to 0^+} \frac{\sin(r^2(\cos^2\theta+\sin^2\theta))}{r^2(\cos^2\theta+\sin^2\theta)}$$

$$= \lim_{r\to 0^+} \frac{\sin(r^2)}{r^2} \to \frac{0}{0} \text{ type}$$

apply L'Hospital's rule

$$\lim_{r\to 0^+} \frac{2r\cos(r^2)}{2r} = \lim_{r\to 0^+} \cos(r^2)$$

$$= \cos(0^2) = 1 \quad \blacksquare$$

ex. Does $\lim_{\vec{x}\to\vec{0}} \frac{x^2-y^2}{x^2+y^2}$ exist?

sol: use polar coords trick:

provided it exists

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{r\to 0^+} \frac{(r\cos\theta)^2-(r\sin\theta)^2}{(r\cos\theta)^2+(r\sin\theta)^2}$$

$$= \lim_{r\to 0^+} \frac{r^2(\cos^2\theta-\sin^2\theta)}{r^2(\cos^2\theta+\sin^2\theta)}$$

$$= \lim_{r\to 0^+} \cos(2\theta)$$

$$= \cos 2\theta$$

notice: θ is present in answer, so answer will be dependent on angle:

if $\theta = \frac{\pi}{2}$, answer will be $\cos(2\cdot\frac{\pi}{2}) = -1$

if $\theta = 0$, answer will be $\lim_{\vec{x}\to\vec{0}} f(x,y) = \cos(0) = 1$

$\therefore$ limit DNE by the curves criterion $\blacksquare$

Continuity

Def'n: a function f is continuous at $\vec{a} \in dom(f)$ when $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$

- A function f is continuous on set D when f is continuous at every $\vec{a} \in D$

ex. every polynomial is continuous everywhere

ex. every rational function is continuous on it's domain

ex. $f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$ is cts everywhere on domain ($(0,0)$ isn't in domain)

ex. $g(x,y) = \begin{cases} \frac{\sin(x^2+y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$ is cts everywhere

remark: the "usual" rules of continuity from calc I still hold

Derivatives of Multivariable Functions

idea is that the derivative measures how a function changes with small changes in it's input in a given direction

defn: The directional derivative of function f of n # of variables at $\vec{a} \in dom(f)$ in the direction of unit vector $\vec{u} \in \mathbb{R}^n$ is

$$D_{\vec{u}} f(\vec{a}) = \lim_{h\to 0^+} \frac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h}$$

ex. compute $D_{\vec{u}} f(\vec{a})$ for $f(x,y) = x\sqrt{y}$ at $\vec{a} = \langle 2,4\rangle$ in direction of $\vec{v} = \langle 2,-1\rangle$

sol. $\vec{u} = \frac{1}{|\vec{v}|}\vec{v} = \frac{1}{\sqrt{5}}\langle 2,-1\rangle = \left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle$

$$D_{\vec{u}} f(\vec{a}) = \lim_{h\to 0^+} \frac{f\left(2+\frac{2}{\sqrt{5}}h,\ 4-\frac{1}{\sqrt{5}}h\right) - f(2,4)}{h}$$

multiply by conjugate

$$= \lim_{h\to 0^+} \frac{\left(2+\frac{2h}{\sqrt{5}}\right)\sqrt{4-\frac{h}{\sqrt{5}}} - 2\cdot\sqrt{4}}{h} = \lim_{h\to 0^+} \frac{\left(2+\frac{2h}{\sqrt{5}}\right)\sqrt{4-\frac{h}{\sqrt{5}}} - 4}{h} \cdot \frac{-\left(2+\frac{2h}{\sqrt{5}}\right)\sqrt{4-\frac{h}{\sqrt{5}}} - 4}{-\left(2+\frac{2h}{\sqrt{3}}\right)\sqrt{4-\frac{h}{\sqrt{5}}} - 4}$$

$$= \lim_{h\to 0^+} \frac{-\left(2+\frac{2h}{\sqrt{5}}\right)\left(4-\frac{h}{\sqrt{5}}\right)+16}{-h\left(4+\left(2+\frac{2h}{\sqrt{5}}\right)\sqrt{4-\frac{1}{\sqrt{5}}h}\right)} = \lim_{h\to 0^+} \frac{-\left(4+\frac{8h}{\sqrt{5}}+\frac{4h^2}{\sqrt{5}}\right)\left(4-\frac{h}{\sqrt{5}}\right)+16}{}$$

$$= \lim_{h\to 0^+} \frac{-\left(16-\frac{4h}{\sqrt{5}}+\frac{32h}{\sqrt{5}}-\frac{8h^2}{5}+\frac{16h^2}{5}-\frac{4}{5\sqrt{5}}h^3\right)+16}{} = \lim_{h\to 0^+} \frac{h\left(\frac{-28}{\sqrt{5}}-\frac{8h}{5}+\frac{4h^2}{5\sqrt{5}}\right)}{-h\left(4+\left(2+\frac{2}{\sqrt{5}}\right)\sqrt{4-\frac{h}{\sqrt{5}}}\right)}$$

$$= \frac{\frac{-28}{\sqrt{5}} - \frac{8\cdot 0}{5} + \frac{4}{5\sqrt{5}}\cdot 0^2}{-\left(4+\left(2+\frac{2}{\sqrt{5}}\cdot 0\right)\sqrt{4-\frac{1}{\sqrt{5}}\cdot 0}\right)} = \frac{-28}{-8\sqrt{5}} = \boxed{\frac{7}{2\sqrt{5}}}$$